Dynamic Linear Models

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Technical paper on prediction using Dynamic linear models

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Abstract

Dynamic linear models were developed in engineering in the early 1960's to monitor and control dynamic systems.Early famous applications have been in the Apollo and Polaris aerospace programs but recently dynamic linear models, and more generally state space models have received an enormous impulse with applications in an extremely vast range of fields, from biology to economics.This paper will apply a dynamic linear model for the task of prediction.More specifically a dynamic linear model will be used to predict the closing price of an exchange traded fund(ETF) called Financial Bull 3X(NYSE symbol: FAS).

1 Introduction

In recent years there has been an increasing interest in the application of state space models to time series analysis.These models consider a time series as the output of a dynamic system with observable and unobservable states perturbed by random disturbances.State space models lend themselves to probabilistic inference and the computations can be implemented as recursive algorithms.In most cases probabilistic inference is carried out by computing the conditional distribution of quantities of interest given the available information and therefore are quite naturally treated within a Bayesian framework.A huge advantage of state space models is that they can be readily applied to univariate or multivariate time series in the presence of non-stationarity and irregular patterns.In this paper, I implement a random walk state space model to predict the closing price for an ETF called Financial Bull 3X(NYSE symbol:FAS).

2 Theory

2.1 The Bayes Rule

The Bayesian approach of learning from experience is done by computing the conditional probability of the event of interest given the available information.Given two events A and B,from the elementary rules of probability the Bayes theorem or the theorem of inverse probability can be derived as

$$P(A|B) = \frac{P(B|A)P(A)}{P(B)}.$$

Let us now apply the Bayesian framework to the problem of statistical inference.Let the observation or the result of a sampling procedure be described by the random vector Y.We would now like to build a parametric model to describe Y,the quantity of interest here is the vector θ of the parameters of the model.In this case Bayesian inference consists of computing the conditional distribution of θ given the observations or the results from sampling.From the Bayes rule this is computed by the formula

$$\pi(\theta|y) = \frac{\pi(y|\theta)\pi(\theta)}{\pi(y)}$$

The conditional distribution $\pi(y|\theta)$ is called the *likelihood*,the distribution $\pi(\theta)$ is called the *prior* and is used to express uncertainity about the parameter vector θ.

2.2 Application of Bayes rule

Let us assume that the observations or the results from sampling can be modeled as

$$Y_t = \theta_t + \epsilon_t \tag{1}$$

where $\epsilon_t \sim \mathcal{N}(0, \sigma^2)$ and are independent and identically distributed(iid).

Now suppose we express our uncertainity about the parameter vector θ as $\theta \sim \mathcal{N}(m_0, C_0)$, where m_0 and C_0 are the prior mean and variance respectively.Uncertainity about the initial guess m_0 can be modeled by choosing a large C_0.Now given the measurements $y_{1:n}$ we can update our belief about the posterior distribution of θ using the Bayes rule.The posterior distribution is given by

$$\pi(\theta|y_{1:n}) \quad \propto \quad likelihood \times prior$$

$$= \prod_{t=1}^{n} \frac{1}{\sqrt{2\pi}\sigma} \exp\{-\frac{1}{2\sigma^2}(y_t - \theta)^2\} \frac{1}{\sqrt{2\pi C_0}} \exp\{-\frac{1}{2C_0}(\theta - m_0)^2\}$$

After some algebra the above expression evaluates to

$$\exp\{-\frac{1}{2\sigma^2 C_0/(nC_0+\sigma^2)}(\theta^2 - 2\frac{nC_0\overline{y}+\sigma^2 m_0}{(nC_0+\sigma^2)}\theta)\}$$

and can be recognized as

$$\theta|y_{1:n} \sim \mathcal{N}(m_n, C_n)$$

where

$$m_n = E(\theta|y_{1:n}) = \frac{C_0}{C_0 + \sigma^2/n}\overline{y} + \frac{\sigma^2/n}{C_0 + \sigma^2/n}m_0 \tag{2}$$

and

$$C_n = Var(\theta|y_{1:n}) = (\frac{n}{\sigma^2} + \frac{1}{C_0})^{-1} \tag{3}$$

The posterior distribution can be computed recursively.We can update the prior $\mathcal{N}(m_{n-1}, C_{n-1})$ on the basis of the observation y_n.Using (2) and (3) we get the resulting posterior as Gaussian with parameters

$$m_n = m_{n-1} + \frac{C_{n-1}}{C_{n-1} + \sigma^2}(y_n - m_{n-1}) \tag{4}$$

and variance

$$C_n = (\frac{1}{\sigma^2} + \frac{1}{C_{n-1}})^{-1} \tag{5}$$

3 The model

In this section we introduce a model for the closing price of the FAS ticker.The model is a simple univariate random walk plus noise model and is defined by

$$Y_t = \mu_t + v_t \qquad v_t \sim \mathcal{N}(0, V) \tag{6}$$

$$\mu_t = \mu_{t-1} + w_t \qquad w \sim \mathcal{N}(0, W) \tag{7}$$

where Y_t is the closing price of FAS at time t and μ_t is the value of the unobservable process at time t.Here the errors V_t and w_t are assumed to be independent within and between themself.With this model in place we are ready to start making predictions about the closing price of FAS.

Let us start with a normal prior $\mathcal{N}(25, 7)$ for $\theta \sim \mathcal{N}(m_0, C_0)$, $\mathcal{N}(0, 1)$ for the v_t's and $\mathcal{N}(0, 14)$ for w_t's.The algorithm for prediction has three steps

- Initial updation step
- Prediction step
- Updating parameters step

Initial Updation step : From the results in section 2 when the first observation comes in at time $t = 1$,we update the parameters of θ_t as

$$m_1 = m_0 + \tfrac{C_0}{C_0+\sigma^2}(Y_1 - m_0)$$

$$C_1^{-1} = \tfrac{1}{\sigma^2} + \tfrac{1}{C_0}$$

Prediction Step : Now we can predict the next closing price of FAS at $t = 2$

based on the the equations (6) and (7).We also find that

$$\theta_2|y_1 \sim \mathcal{N}(a_2, R_2)$$

where

$$a_2 = E(\theta_1 + v + w_2|y_1) = m_1 + v$$

and variance

$$R_2 = Var(\theta_1 + v + w_2|y_1) = C_1 + \sigma_w^2$$

We can also predict the next closing price of FAS (y_2) given y_1.Based on equation (5) we find that

$$Y_2|y_1 \sim \mathcal{N}(f_2, Q_2)$$

where

$$f_2 = E(\theta_2 + \epsilon_2|y_1) = a_2$$

and

$$Q_2 = Var(\theta_2 + \epsilon_2|y_1) = R_2 + \sigma^2$$

Updating parameters : Now at time $t = 2$ the new closing price Y_2 becomes available.We can now update the parameters for θ by computing the posterior distribution $\theta_2|y_{1:2}$.Thus by Bayes formula we get

$$\theta_2|y_1, y_2 \sim (m_2, C_2)$$

where

$$m_2 = a_2 + \frac{R_2}{R_2+\sigma^2}(y_2 - f_2)$$

and

$$C_2 = \frac{\sigma^2 R_2}{\sigma^2+R_2}$$

with the role of the prior being played by the density $\mathcal{N}(a_2, R_2)$ and the likelihood is given by the density of Y_2 given (θ_2, y_1).Since Y_2 is independent from the past observations given θ_2 the likelihood is given by

$$Y_2|\theta_2 \sim \mathcal{N}(\theta_2, \sigma^2)$$

4 Results

The algorithm described in the sections above was implemented in R and used to predict the closing price of FAS.The closing prices for 255 days were predicted and at the end of the day the parameters for making future pre-

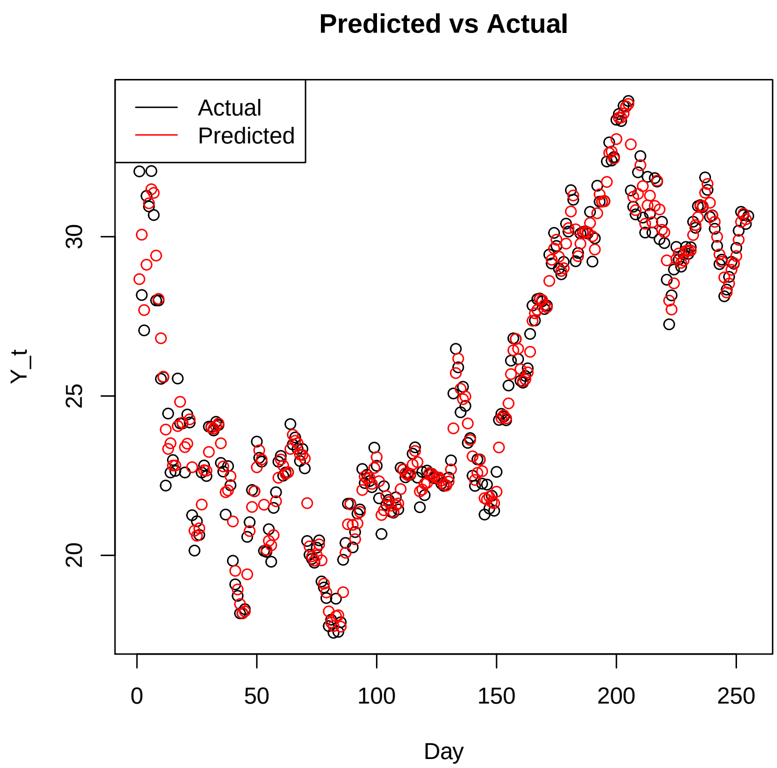

Figure 1: Predicted vs Actual Closing prices of FAS.

dictions was updated.The results of the simulations are below and are self explanatory.

5 Conclusion and Future work

From the results it can be seen that this method is quite good at making predictions of the next days closing price of FAS.In order words, the method seems to work well for making price predictions for time $(t+1)$ given data upto time t.Predictions for time $(t+n)$ where n is a non zero integer can be made based on the equations that specify the dynamics of the system.However, as n increases these predictions will get worse.One area of future work is to be able to make long term predictions without a significant decrease in the prediction

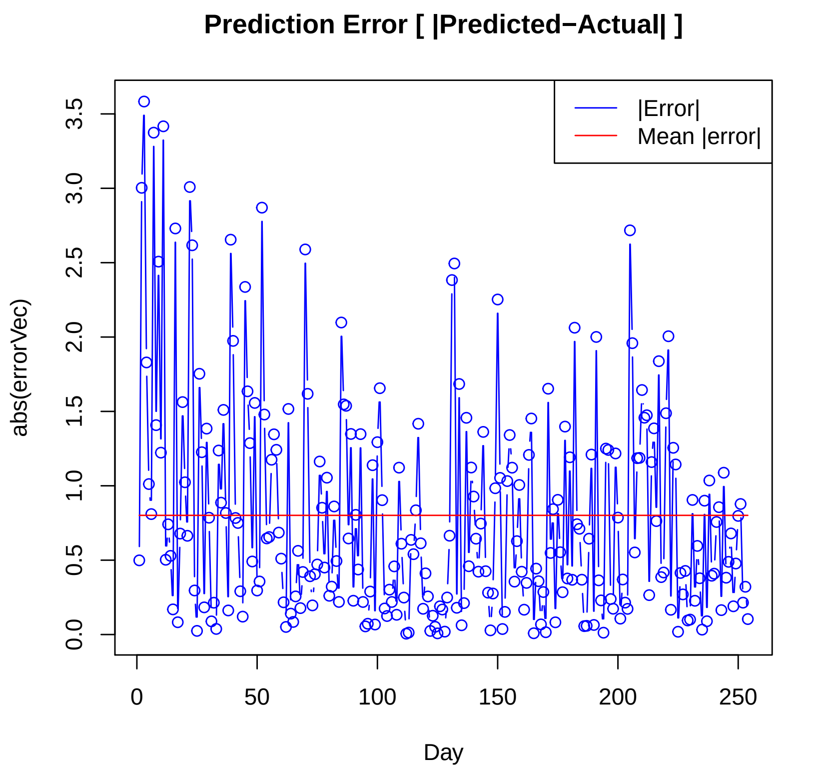

Figure 2: Prediction Error and mean of the prediction error

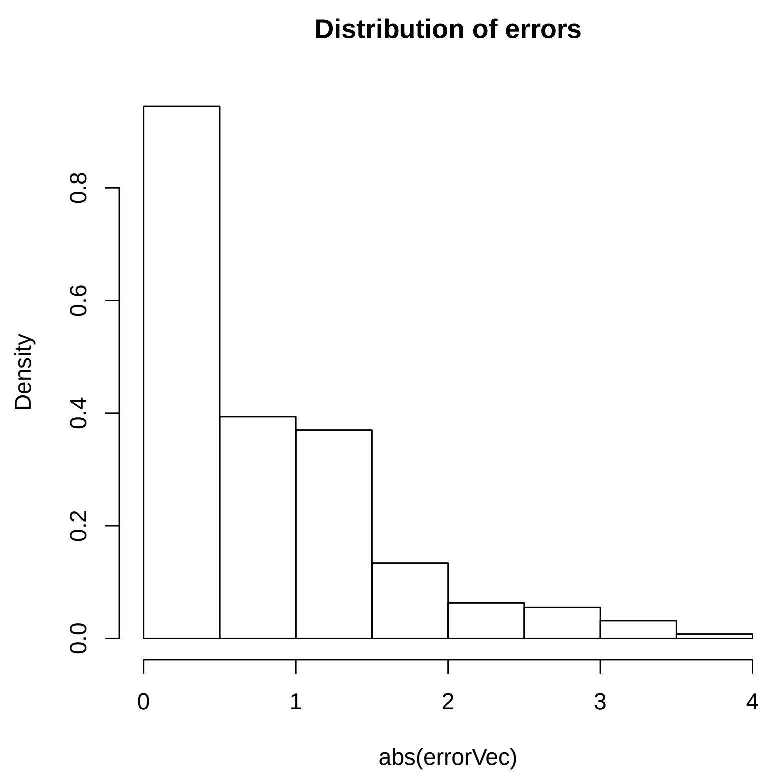

Figure 3: Distribution of the prediction errors

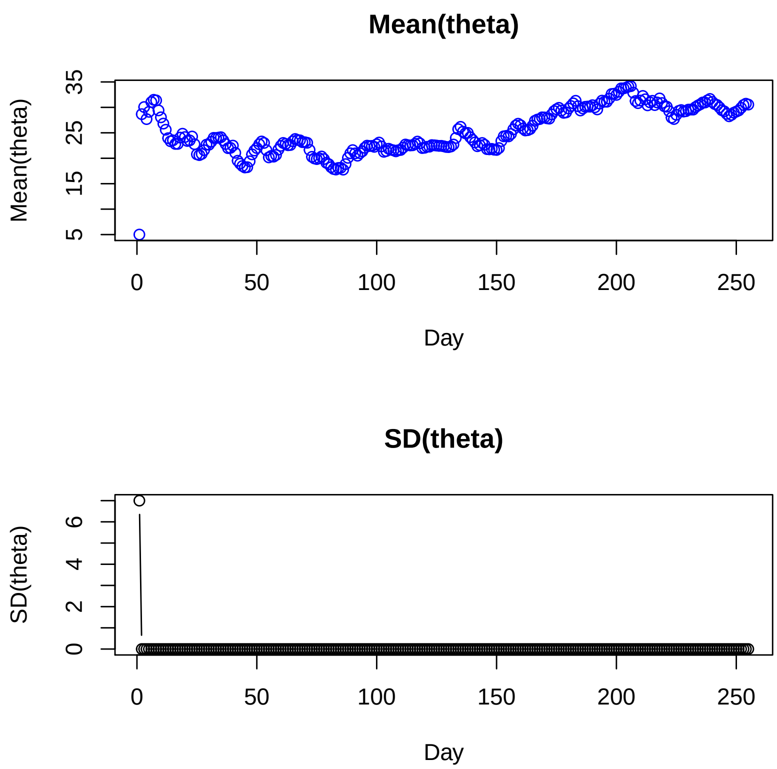

Figure 4: Evolution of the mean and s.d for θ

power.Another area of future work can be to predict intraday prices of FAS and build a high frequency trading agent that can trade based on these predictions in realtime.The problem though would have to be reformulated as finding the parameters (θ) and then picking an action (a) at time (t) that maximizes the conditional expected profit given the observed price of FAS,the fixed transaction cost and the predicted value of FAS at time ($t+n$) $[E(P|y_{1:t}, a_t, C, y_{pred_{(t+n)}})]$.

The model implemented here is a simple random walk model,going forward we also plan to apply this method to more complex stochastic volatility models with the hope of modeling the market dynamics and price changes as close as possible to reality.

6 References

1. Petris, Petrone, and Campagnoli. Dynamic Linear Models with R. Springer (2009).

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